Fermi liquid theory notes

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The basic premise of Landau's Fermi-liquid theory is adiabatic continuity - that the particle interaction can be adiabatically turned on or off. Therefore states of an interacting system (Fermi liquid) can be mapped back to states of an non-interacting system (Fermi gas) with well-defined momentum. In other words particle states are completely specified by the adiabatic-equivalent non-interacting particle momentum.

Note that the momentum is only a good quantum number in the non-interacting system and it is not very meaningful to talk about momentum in the interacting system. As such whenever we mention momentum in this note we always refer to the adiabatic-equivalent non-interacting momentum. We will also follow the notations in [1]

Calculation of speed of sound

The speed of sound c is given by

$$c^2 = \frac{\partial P}{\partial \rho} = \frac{\partial P}{\partial (mN/\Omega)} = \frac{\Omega}{m} \frac{\partial P}{\partial N} \tag{1}$$

where P is the pressure, ρ is the mass density, m is the mass of a single particle, N is the total number of particles and Ω is the total volume.
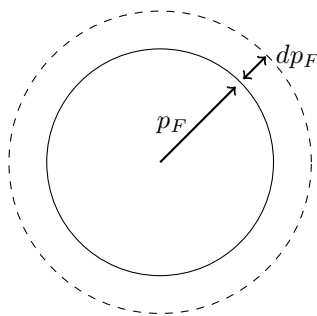

Figure 1: The Fermi momentum increases by dp_F when the chemical potential increases by $d\mu$.

The free energy is given by $dF = -Pd\Omega - SdT + \mu dN$. Using the triple product rule and Maxwell's relations we have

$$\frac{\partial \mu}{\partial N} = -\frac{\partial \Omega}{\partial N}\frac{\partial \mu}{\partial \Omega} = -\frac{\partial \Omega}{\partial N}\frac{\partial}{\partial \Omega}\left(\frac{\partial F}{\partial N}\right) = \frac{\Omega}{N}\frac{\partial P}{\partial N} \tag{2}$$

Thus it all boils down to finding $\frac{\partial \mu}{\partial N}$ or equivalently $\frac{\partial N}{\partial \mu}$. To do so we consider momentum space of the system and look at how the Fermi surface changes when the chemical potential increases. The situation is illustrated in figure 1.

We can write down the chemical potential before and after in terms of quasi-particle energies:

$$\mu + d\mu = \epsilon_{p_F + dp_F}(\mu + d\mu), \ \mu = \epsilon_{p_F}(\mu) \tag{3}$$

where $\epsilon_p(\mu)$ denotes the energy of the quasi-particle with momentum p when the chemical potential is μ. We can take the difference and obtain

$$d\mu = \epsilon_{p_F + dp_F}(\mu + d\mu) - \epsilon_{p_F}(\mu) \tag{4}$$

$$= \epsilon_{p_F + dp_F}(\mu + d\mu) - \epsilon_{p_F}(\mu + d\mu) + \epsilon_{p_F}(\mu + d\mu) - \epsilon_{p_F}(\mu) \tag{5}$$

$$= (\nabla_p \epsilon_p)|_{p_F} dp_F + \sum_{p'} f_{p_F p'} \delta n_{p'} \tag{6}$$

The derivative of energy with respect to momentum at the Fermi level is defined as the Fermi velocity $v_F = p_F/m^*$ where m^* is the effective mass. The second term is the additional internal energy creating by the increase in particle occupancy $\delta n_{p'}$ and $f_{pp'}$ is the Landau parameter. From figure 1 we can see that the added particles reside within a thin shell above the original Fermi level with thicken dp_F, thus we can write

$$\delta n_{p'} = \delta(p' - p_F) dp_F \tag{7}$$

We can express this in terms of the density of states at Fermi surface $\nu(0) = \sum_p \delta(\epsilon_p - \mu)$ by converting the delta function via

$$\delta(\epsilon_p - \mu) = \frac{\delta(p - p_F)}{|\nabla_p \epsilon_p|} = \frac{\delta(p - p_F)}{v_F} \tag{8}$$

Thus,

$$d\mu = v_F dp_F + \sum_{p'} f_{p_F p'}\left(\delta(\epsilon_p - \mu)v_F dp_F\right) \tag{9}$$

Since the term in the bracket is isotropic and spin-independent, only the isotropic component of $f_{p_F p'}$ contributes and furthermore it can be replaced by its spin-average:

$$d\mu = v_F dp_F + \sum_{p'} f_0^S \delta(\epsilon_p - \mu)v_F dp_F \tag{10}$$

$$= v_F dp_F + f_0^S(\sum_{p'} \delta(\epsilon_p - \mu))v_F dp_F \tag{11}$$

$$= (1 + F_0^S)v_F dp_F \tag{12}$$

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where $F_0^S = f_0^S \nu(0)$.

We can now calculate the change in total number of particles by summing over $\delta n_p'$:

$$dN = \sum_{p'} \delta n_{p'} = v_F dp_F \left(\sum_{p'} \delta(\epsilon_p - \mu) \right) = \frac{d\mu}{1 + F_0^S} \nu(0) \tag{13}$$

$$\frac{dN}{d\mu} = \frac{\nu(0)}{1 + F_0^S} \tag{14}$$

To calculate $\nu(0)$, we need to consider the sum over the delta function:

$$v_F \nu(0) = \sum_p \delta(p - p_F) \tag{15}$$

$$= \frac{d}{dp_F} \sum_p \Theta(p_F - p) \tag{16}$$

$$= \frac{d}{dp_F} \frac{2 \times 4\pi p_F^3/3}{h^3/\Omega} \tag{17}$$

$$= \frac{\Omega m^* p_F}{\pi \hbar^3} \tag{18}$$

The factor of h^3/Ω is due to quantization of the phase space. The extra factor of 2 is because we are summing over both spins.

Combining everything, we have

$$c^2 = \frac{p_F}{3mm^*}(1 + F_0^S) \tag{19}$$

Calculation of magnetic susceptibility

The calculation of magnetic susceptibility follows a similar approach. In momentum space the Fermi surfaces of the spin-up and spin-down particles will be shifted in opposite directions but by the same amount dp_F. The chemical potentials of the two spins are equal to each other, and also to the original chemical potential since μ can at most depend on B^2 as B is a vector quantity and μ is scalar.

We can write down the shift in the chemical potential for each spin:

$$d\mu = \epsilon_{p_F + \sigma dp_F}(B) - \epsilon_{p_F}(0) \tag{20}$$

$$0 = v_F \sigma dp_F - \mu_B \sigma B + \sum_{p'\sigma'} f_{p_F \sigma p' \sigma'} \delta(p' - p_F) \sigma' dp_F \tag{21}$$

$$= v_F \sigma dp_F - \mu_B \sigma B + \sum_{p'} (f_{p_F \sigma p' \sigma} - f_{p_F \sigma p' - \sigma}) \delta(p' - p_F) \sigma dp_F \tag{22}$$

$$= v_F \sigma dp_F - \mu_B \sigma B + 2 f_0^A \left(\frac{\nu(0)}{2} \right) \sigma dp_F \tag{23}$$

$$v_F dp_F = \frac{\mu_B B}{1 + F_0^A} \tag{24}$$

where we divide $\nu(0)$ by 2 because we are summing over just one spin. $\sigma = \pm 1$ denotes the direction of the spin.

The total spin is given by

$$s = dn_+ - dn_- = \frac{\nu(0)}{2}v_F dp_F - \frac{\nu(0)}{2}v_F(-dp_F) = \nu(0)v_F dp_F \qquad (25)$$

Combining everything we have

$$\chi = \frac{\mu_B s}{\Omega B} = \frac{\nu(0)}{\Omega}\frac{\mu_B^2}{1 + F_0^A} \qquad (26)$$

$$= \frac{m^* p_F}{\pi^2 \hbar}\frac{\mu_B^2}{1 + F_0^A} \qquad (27)$$

References

[1] Pines, David. Theory of Quantum Liquids: Normal Fermi Liquids. CRC Press, 2018.